On Generalized Perfect Rings

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(Joint work with D. Herbera)

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Definitions[A. Amini,B. Amini, Ershad, Sharif-2007]

Let R be an associative ring with 1. All modules are unital. Ring homomorphisms preserve 1.

- Let F and M be right R-modules such that F_R is flat. A module epimorphism $f\colon F \to M$ is said to be a *G-flat cover* of M if $\mathrm{Ker}\,(f)$ is a small submodule of F.

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- A ring R is called *right generalized perfect* (right G-perfect, for short) if every right R-module has a G-flat cover.
- A ring R is called *G-perfect* if it is both left and right G-perfect.

- $\{$ perfect rings $\} \subseteq \{$ G-perfect rings $\}$
- $\{$ Von Neumann regular rings $\} \subseteq \{$ G-perfect rings $\}$
- $\{$ G-perfect rings $\}$ is closed under finite products and quotients.

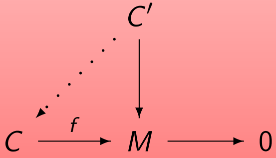
Definition (due to Auslander and Enochs)

Let $\mathcal{C}$ be a class of right R-modules, and let M_R be a right R-module.

A module homomorphism $f\colon C \to M$ is a $\mathcal{C}$-precover of M if it satisfies that

(i) $C \in \mathcal{C}$;

(ii) any diagram with $C' \in \mathcal{C}$



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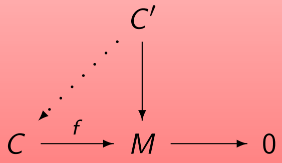

can be completed to a commutative diagram.

The homomorphism f is a $\mathcal{C}$-cover if, in addition, it is right minimal.

Recall that $f\colon C \to M$ is said to be right minimal if for any $g \in \mathrm{End}_R(C)$, $f = fg$ implies g bijective

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- In the case of perfect rings projective covers, flat covers and G-flat covers coincide.
- In the case of von Neumann regular rings flat covers are G-flat covers

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- Kernel of any flat cover is a cotorsion module.
- Any M_R fits into an exact sequence

$$0 \to B \to L \xrightarrow{g} M$$

  where L is flat and B is cotorsion. g is a flat precover.

Example due to A. Amini,B. Amini, Ershad, Sharif-2007

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- **Case 1** $Soc(E/M) = 0$. $\pi : E \to E/M$ and $i : E/M \to E/M$ are both G-flat covers of E/M. But $E \ncong E/M$.

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- **Case 2** $Soc(E/M) \neq 0$. There is $K_R \subseteq E_R$ such that K/M is a simple R-module. $\pi : K \to K/M$ and $i : K/M \to K/M$ are both G-flat covers of K/M. But $K \ncong K/M$.

Some results from A. Amini,B. Amini, Ershad, Sharif-2007

- R is right G-perfect $\Longrightarrow J(R)$ is right T-nilpotent.
- R is right duo and right G-perfect $\Longrightarrow R/J(R)$ is von Neumann regular.

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Conjecture: R is right G-perfect $\Longrightarrow$ semiregular ???
Our Answer: No!!!

Basic Definitions

A pair $(\mathcal{X}, \mathcal{Y})$ of subclasses of $\mathrm{Mod}\text{-}R$ is said to be a torsion pair if

(i) $\mathrm{Hom}_R(X, Y) = \{0\}$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(ii) If X_R is a right R-module such that $\mathrm{Hom}_R(X, Y) = \{0\}$ for any $Y \in \mathcal{Y}$ then $X \in \mathcal{X}$.

(iii) If Y_R is a right R-module such that $\mathrm{Hom}_R(X, Y) = \{0\}$ for any $X \in \mathcal{X}$ then $Y \in \mathcal{Y}$.

In this case, $\mathcal{X}$ is said to be a torsion class and $\mathcal{Y}$ is a torsion-free class. The objects of $\mathcal{X}$ are called torsion modules and the objects in $\mathcal{Y}$ are called torsion-free modules.

Basic Definitions

Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair. If M_R is a right R-module, the largest submodule of M_R that is an object of $\mathcal{X}$ called the torsion submodule of M and is denoted by $t(M)$. t is indeed a functor and a radical. So that, there is an exact sequece

$$0 \to t(M) \to M \to M/t(M) \to 0$$

where $M/t(M) \in \mathcal{Y}$.

Basic Definitions

- A class of modules $\mathcal{X}$ is torsion if and only if it is closed under isomorphisms, extensions, coproducts and quotients.
- Dually, a class of modules $\mathcal{Y}$ is a torsion-free class if it is closed under isomorphism, extensions, submodules and products.

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- Notice that if a class of modules $\mathcal{Y}$ is closed by products, coproducts, subobjects, quotients and extensions then $\mathcal{Y}$ is a torsion class and a torsion free class at the same time. Therefore, one has a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ such that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. Such a triple is called a TTF-triple.

Let $0 \longrightarrow M \xrightarrow{h} N \xrightarrow{f} K \longrightarrow 0$ be an exact sequence of right R-modules and let $L \xrightarrow{g} K \longrightarrow 0$ be an onto homomorphism. We consider the pullback of f and g to obtain a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X & = & X = Kerg & & \\
 & & & & \downarrow^{\varepsilon_2} & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{\varepsilon_1} & L' & \xrightarrow{\pi_2} & L & \longrightarrow & 0 \\
 & & \| & & \downarrow^{\pi_1} & & \downarrow^{g} & & \\
0 & \longrightarrow & M & \xrightarrow[h]{} & N & \xrightarrow[f]{} & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\qquad (1)
$$

In (1),

- $L' = \{(x, y) \in N \oplus L | f(x) = g(y)\}$.
- The maps $\pi_1 \colon L' \to N$ and $\pi_2 \colon L' \to L$ are restrictions of the canonical projections $\pi_1 \colon N \oplus L \to N$ and $\pi_2 \colon N \oplus L \to L$, respectively.
- The homomorphism $\varepsilon_1 \colon M \to L'$ is defined by $\varepsilon_1(x) = (h(x), 0)$ for each $x \in M$, and $\varepsilon_2 \colon X \to L'$ is defined by $\varepsilon_2(y) = (0, y)$ for each $y \in X$.

Lemma[A, Herbera-2016]

Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in Mod-R such that the associated torsion radical t is exact. Assume that in diagram (1), $M \in \mathcal{X}$ and K, $L \in \mathcal{Y}$.

- If X is small in L, then $\varepsilon_2(X)$ is small in L'.
- In particular, if L_R and M_R are flat, then $\pi_1 \colon L' \to N$ is a G-flat cover of N.
- g is right minimal if and only if π_1 is right minimal.

Useful facts on TTF-triples

Let R and S be rings such that there is an exact sequence

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where φ is a ring morphism such that ${}_R S$ becomes a flat module. Consider the following classes of modules

$$\mathcal{X} = \{X \in \mathrm{Mod}\text{-}R \mid XI = X\}$$

$$\mathcal{Y} = \{Y \in \mathrm{Mod}\text{-}R \mid YI = \{0\}\}$$

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then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF-triple such that the torsion pair $(\mathcal{X}, \mathcal{Y})$ is hereditary and $\mathrm{Ext}^i_R(X, Y) = 0$ for any $i \geq 0$, $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

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then $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF-triple such that the torsion pair $(\mathcal{X}, \mathcal{Y})$ is hereditary and $\mathrm{Ext}^i_R(X, Y) = 0$ for any $i \geq 0$, $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Moreover, the torsion radical associated to the torsion class $\mathcal{X}$ is naturally equivalent to the exact functor $- \otimes_R I$, and the torsion radical associated to the class $\mathcal{Y}$ is naturally equivalent to the functor $\mathrm{Hom}_R(S, -)$.

Corollary

Let R and S be rings such that there is an exact sequence

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where φ is a ring morphism such that S becomes a flat R-module on the right and on the left. Then:

(i) M_R is flat if and only if $M \otimes_R S$ is a flat right S-module and MI is a flat right R-module.

(ii) Let M be a right S-module, then M is cotorsion as a right R-module if and only if it is cotorsion as an S-module.

Proposition[A, Herbera-2016]

Let $S \subseteq T$ be an extension of rings. Let

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}.$$

Then, the following statements hold.

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Then, the following statements hold.

(i) The map $\varphi \colon R \to S$ defined by $\varphi(x_1, x_2, \ldots, x_n, x, x, \ldots) = x$ is a ring homomorphism with kernel

$$I = \bigoplus_{\mathbb{N}} T = \bigoplus_{i \in \mathbb{N}} e_i R,$$

where $e_i = (0, \ldots, 0, 1^{(i)}, 0, 0, \ldots)$ for any $i \in \mathbb{N}$.

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(iii) For any $i \in \mathbb{N}$, the canonical projection into the i-th component $\pi_i\colon R \to T$ has kernel $(1 - e_i)R$ so that T is projective as a right and as a left R-module via the R-module structure induced by π_i.

Remark

Let R be a ring as in the Proposition. Then there is a TTF-triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ associated to the pure exact sequence

$$0 \to I \to R \xrightarrow{\varphi} S \to 0$$

where $\mathcal{X} = \{X \in \mathrm{Mod}-R \mid X = \oplus_{i \in \mathbb{N}} Xe_i\}$,
$\mathcal{Y} = \{Y \in \mathrm{Mod}-R \mid YI = \{0\}\}$
$\mathcal{Z} = \{Z \in \mathrm{Mod}-R \mid \mathrm{ann}_Z(I) = \{0\}\}$. Also, for any $i \in \mathbb{N}$, the split sequence

$$0 \to R(1-e_i) \to R \xrightarrow{\pi_i} T \to 0$$

yields a corresponding (split) TTF-triple $(\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i)$.

Proposition[A, Herbera-2016]

(i) $J(R)$ contains $J = \bigoplus_{\mathbb{N}} J(T)$. Moreover, J is essential on both sides into $J(R)$. In particular, $J(R) = 0$ if and only if $J(T) = 0$.

(ii) R is von Neumann regular if and only if S and T are von Neumann regular.

(iii) Let M_R be a right R-module. Then M_R is flat if and only if $M \otimes_R S$ is a flat right S-module and, for any $i \in \mathbb{N}$, Me_i is a flat right T-module.

Main Theorem [A, Herbera-2016]

Let $S \subseteq T$ be an extension of rings. Assume T is von Neumann regular and that S is right G-perfect. Then

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}$$

is a right G-perfect ring such that $J(R) = 0$.
Moreover, if S is a ring such that flat covers are G-flat covers, then also R satisfies this property.

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- Since T is von Neumann regular, for any $i \in \mathbb{N}$, Ne_i is a flat T-module.
- Hence NI is flat as a right R-module.

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Let $0 \to X \to L \xrightarrow{h} N/NI \to 0$ be a G-flat cover of the right S-module N/NI. Considering the pullback of h and f yields the following diagram with exact rows and columns

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\begin{array}{ccccccccc}
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 & & & & \downarrow & & \downarrow & & \\
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0 & \longrightarrow & NI & \longrightarrow & L' & \xrightarrow{\pi_2} & L & \longrightarrow & 0 \\
 & & \parallel & & \downarrow^{\pi_1} & & \downarrow^{h} & & \\
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- X_R is also a cotorsion module, hence $0 \to X \to L' \xrightarrow{\pi_1} N \to 0$ is a flat precover of N.

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- it follows that π_1 is also a flat cover.

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- If $\dim_F(S) = n$, then $S \hookrightarrow T = \mathbb{M}_n(F)$ which is von Neumann regular.
- Therefore,

$$R = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T, x \in S\}$$

is G-perfect by Main Theorem.

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  is G-perfect by Main Theorem.
- By the properties of the ring, $J(R) = 0$ and R is not von Neumann regular.

For a particular realization of such a ring R consider, for example, $S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. In this case, T can be taken to be $M_2(F)$.

Example 2 [A, Herbera-2016]

Let R be as in Example (1).

- Then, $R \subseteq \prod \mathbb{M}_n(F) = T'$ which is a von Neumann regular ring.
- $R' = \{(x_1, x_2, \ldots, x_n, x, x, \ldots) | n \in \mathbb{N}, x_i \in T', x \in R\}$ is also a G-perfect ring.

Open Questions

In general, it is difficult to compute Enochs flat covers. If projective covers exist, then they coincide with Enochs flat covers. So the question is:

Question 1: What is the relation, if any, between G-flat covers and Enochs flat covers?

Question 2: Let R be a semiregular ring with right T-nilpotent Jacobson radical, is it G-perfect?